Review for Final Exam

Limits and continuity of functions of two and three variables. For example, In order for a limit like $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist, it must be the case that there exists $L \in \mathbb{R}$ such that $|f(x,y) - L|$ tends to zero as $||(x,y) - (a,b)||$ tends to zero.

In particular, the limit must exist along any path approaching the point (a,b).

$f(x,y)$ is continuous at the point (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. That is, the limit should exist and the limit should be the expected value $f(a,b)$.

For functions of two or three variables, you should be able to show that limits exist or do not exist and be able to verify whether or not a multivariable function is continuous at a point.

Partial derivatives, directional derivatives, gradients, derivatives, tangent planes, chain rule.

(a) If we fix a unit vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and a point (a, b, c) in $\mathbb{R}^3$, the directional derivative of $f(x, y, z)$ at $P = (a, b, c)$ in the direction of $\vec{u}$ is given by

$$\lim_{t \to 0} \frac{f(P + t\vec{u}) - f(P)}{t} = \lim_{t \to 0} \frac{f(a + tu_1, b + tu_2, c + tu_3) - f(a, b, c)}{t}$$

To obtain the partial derivatives of f with respect to x, y or z, one simply takes $\vec{u}$ to be $\vec{i}, \vec{j}$ or $\vec{k}$.

To calculate the partials of f with respect to x, y or z, one differentiates f with respect to the corresponding variable, treating all other variables as constants.

(b) To calculate the directional derivative of $f(x, y, z)$ at the point $P = (a, b, c)$ in the direction of the unit vector $\vec{u}$, one takes $\nabla f(a, b, c) \cdot \vec{u}$, where $\nabla f(a, b, c)$ is the gradient of f at $P = (a, b, c)$.

(c) Recall that the gradient of f at the point P is the vector $\langle \frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P), \frac{\partial f}{\partial z}(P) \rangle$ and indicates the direction of greatest rate of change of f at P.

In other words, the directional derivative of f at P is greatest, when $\vec{u}$ points in the direction of $\nabla f(P)$. That maximal rate of change is given by the length of the gradient vector, $||\nabla f(P)||$. Note that $-\nabla f(P)$ points in the direction where the rate of change of $f(P)$ is the least.

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(d) For $f(x, y)$ we may form the linear function

$$L(x, y) = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + f(a, b).$$

Note that, when f has continuous first order partial derivatives at (a, b), then $L(x, y)$ is also the tangent plane to the graph of $z = f(x, y)$ at the point $(a, b, f(a, b))$.

(e) Heuristically, $f(x, y)$ is differentiable at (a, b) if the linear function $L(x, y)$ above is a good approximation to $f(x, y)$ near (a, b). This can be made precise using limits. More precisely, $f(x, y)$ is differentiable at the point (a, b) if the linear function $L(x, y)$ exists and

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - L(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

Geometrically speaking this means that there is a well-defined tangent plane to the graph of $f(x, y)$ at $(a, b, f(a, b))$.

The same notions of partial derivatives and differentiability apply to functions of three or more variables.

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Important points are the following:

(i) Differentiability implies continuity;

(ii) $f(x, y)$ is differentiable at (a, b) if it has continuous first order partials at (a, b);

(iii) If $f(x, y)$ is differentiable at (a, b), then the directional derivative of $f(x, y)$ at (a, b) exists in all directions;

(iv) If $f(x, y)$ is differentiable at (a, b), then the linear function $L(x, y)$ in the definition is the linear function describing the tangent plane to the graph of $f(x, y)$ at $(a, b, f(a, b))$.

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(f) For a surface given implicitly as $f(x, y, z) = k$ (i.e., a *level surface*), the tangent plane to the surface at (a, b, c) is given by

$$\frac{\partial f}{\partial x}(a, b, c)(x - a) + \frac{\partial f}{\partial x}(a, b, c)(y - b) + \frac{\partial f}{\partial z}(a, b, c)(z - c) = 0,$$

since the gradient vector $\nabla f(a, b, c)$ is normal to the surface at (a, b, c).

(g) Chain rule: For a function $f(x_1, \ldots, x_n)$ of n variables, if each $x_j = x_j(u_1, \ldots, u_r)$ is a function of r variables, then we may differentiate f with respect to each u_j. Upon doing so, we get

$$\frac{\partial f}{\partial u_j} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial u_j} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial u_j} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial u_j}.$$

(h) Recall that the mixed second order partial derivatives like $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are equal if the relevant first and second order partial derivatives are continuous.

(i) You should be able to calculate any of the derivatives above using the limit definition. You should also be able to use the limit definitions to determine if certain partial derivatives are continuous or any of the properties above fail.

Relative extreme values for functions of two variables, and constrained maxima and minima for functions of two or three variables.

You should be able to identify critical points and classify them for functions of two or thee variables. You should be able to find extreme values of constrained equations using Lagrange multipliers, including extreme values on regions in $\mathbb{R}^2$ with a boundary.

(a) For a function $f(x, y)$ of two variables, its critical points are the ordered pairs (a, b) that satisfy the system of equations $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$. One then applies the second derivative test to determine if a given critical point yields a relative maximum value, a relative minimum value, or a saddle point.

Second Derivative Test: Let (a, b) be a critical point of $f(x, y)$. The *discriminant* at (a, b) is given by

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2.$$

Assuming $f(x, y)$ has continuous partial derivatives, we have:

(i) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a relative minimum.

(ii) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a relative maximum.

(iii) If $D < 0$, $(a, b, f(a, b))$ is a saddle point.

(iv) If $D = 0$, the test is inconclusive.

(b) Lagrange multipliers. If we wish to maximize or minimize $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, we solve $\nabla f = \lambda \nabla g$, subject to the constraint. i.e., we find solutions to the system of equations:

$$
\begin{aligned}
f_x(x, y, z) &= \lambda g_x(x, y, z) \\
f_y(x, y, z) &= \lambda g_y(x, y, z) \\
f_z(x, y, z) &= \lambda g_z(x, y, z) \\
g(x, y, z) &= k
\end{aligned}
$$

and then test them in $f(x, y, z)$.

The largest and smallest resulting values are the maximum and minimum values sought. This method can be used to find max and min values of F on a closed and bounded domain.

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Parametrized curves and surfaces. A parametrized curve C in 3-space is given by $C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

You should know the parametrizations for basic curves such as straight lines, circles, ellipses, etc.

The tangent vectors are given by $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$.

If P, Q are points on the curve such that $\vec{r}(t_1) = P$ and $\vec{r}(t_2) = Q$, then the length of the curve from P to Q is given by $\int_{t_1}^{t_2} ||\vec{r}\,'(t)||\; dt$.

This requires that $\vec{r}(t)$ does not trace over itself (i.e., it is a 1-1 function of t) and $\vec{r}\,'(t)$ is continuous.

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A parametrized surface S in 3-space is given by $G(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, with $(u,v) \in D$, the domain of $G(u,v)$. Note the points of S live in xyz-space and D is contained in the uv-plane.

One should know parametrizations for basic surfaces like planes, spheres, ellipsoids, etc.

For a point $P = G(u_0, v_0)$, $\pm\,(\mathbf{T}_u \times \mathbf{T}_v)(u_0, v_0)$ are normal vectors to S at P. It follows that

$$(\mathbf{T}_u \times \mathbf{T}_v)(u_0, v_0) \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

is the equation of the plane tangent to S at $P = G(u_0, v_0) = (x_0, y_0, z_0)$.

The surface area of S is $\int\int_D ||\mathbf{T}_u \times T_v||\; dA$.

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Two important parameterizations of surfaces are the following:

(i) If S is the graph of $z = f(x, y)$, with $(x, y) \in D \subseteq \mathbb{R}^2$, then $G(u, v) = \langle u, v, f(u, v) \rangle$, with $(u, v) \in D$ is a parametrization of S.

(ii) If S is the sphere of radius R centered at the origin in 3-space,then the following is a standard parametrization of S: $G(\phi, \theta) = (R \sin(\phi) \sin(\theta), R \sin(\phi) \cos(\theta), R \cos(\phi))$, with $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Note that

$$\mathbf{T}_\phi \times \mathbf{T}_\theta = R^2 \sin(\phi)(\sin(\phi) \sin(\theta), \sin(\phi) \cos(\theta), \cos(\phi)).$$

Double and triple integrals. Double or triple integrals arise when, in the first case, we wish to integrate a function $f(x, y)$ over a flat region R in the xy-plane, and in the second case, when we wish to integrate a function $f(x, y, z)$ over a solid region in 3-space.

For continuous functions, and well-behaved domains of integration, these integrals can be calculated as iterated integrals, by Fubini's Theorem.

For double integrals, we can integrate over regions R of either Type I: $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$, in which case, $\int\int_R f(x, y)\ dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\ dy\ dx$

or a region R of Type II: $c \leq y \leq d$ and $g(y) \leq x \leq h(y)$, in which case $\int\int_R f(x, y)\ dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\ dx\ dy$.

Many regions are of both types and switching the order of integration over regions of this type may render a previously difficult or unmanageable integral manageable.

Another technique for solving a difficult double integral is to use the Change of Variables Theorem.

Given $\int\int_R f(x,y)\ dA$, let $G(u,v) = (x(u,v), y(u,v))$ be a transformation from the uv-plane to the xy-plane (with suitable differentiability conditions). Let $|\frac{\partial(x,y}{\partial(u,v)}|$ denote the absolute value of the determinant of the matrix $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$.

The determinant in question is called the Jacobian of *G(u,v)*. Then

$$\int\int_R f(x,y)\ dA = \int\int_D f(x(u,v), y(u,v)) \cdot \left|\frac{\partial(x,y}{\partial(u,v)}\right|\ dA,$$

where D is the domain of G and $G(D) = R$.

A special case is the case of polar coordinates, where

$$G(r,\theta) = (r\cos(\theta), r\sin(\theta)) \text{ and } \left|\frac{\partial(x,y}{\partial(r,\theta)}\right| = r.$$

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For triple integrals, it is possible to get six fundamental regions, but of types similar to those above.

For example, if a region B in 3-space is defined by the inequalities: $a \le x \le b$, $g(x) \le y \le h(x)$, and $s(x, y) \le z \le t(x, y)$,we have

$$\int\int\int_B f(x, y, z)\ dV = \int_a^b \int_{g(x)}^{h(x)} \int_{s(x,y)}^{t(x,y)} f(x, y, z)\ dz\ dy\ dx.$$

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The Change of Variables Theorem also has a 3-dimensional version: We start with a transformation

$$G(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$$

(with suitable differentiability conditions) and its corresponding Jacobian, i.e., $\frac{\partial(x,y,z)}{\partial(u,v,w)}$, which is the determinant of the matrix

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

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If B_0 is the domain of G in uvw-space, with $G(B_0) = B$, then

$$\int\int\int_B f(x, y, z)\ dV =$$

$$\int\int\int_{B_0} f(x(u, v, w), y(u, v, w), z(u, v, w)) \cdot \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|\ dV.$$

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Some important special case are:

Cylindrical coordinates: $G(r, \theta, z) = (r \cdot \cos(\theta), r \cdot \sin(\theta), z)$, with $0 \leq \theta \leq 2\pi$, $r \geq 0$, and $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$.

Spherical coordinates:
$T(\rho, \theta, \phi) = (\rho \cdot \sin(\theta)\sin(\phi), \rho \cdot \cos(\theta)\sin(\phi), \rho \cdot \cos(\phi))$, with $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2 \cdot \sin(\phi)$.

Linear Transformations:

$$T(u, v, w) = (au + dv + gw, bu + ev + hw, cu + fv + iw),$$

which take the unit cube in uvw-space to the parallelepiped spanned by the vectors $(a, b, c), (d, e, f), (g, h, i)$. In this case,

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = a(ei - fh) - d(bi - ch) + g(bf - ce).$$

Translations: $T(u, v, w) = (u + a, v + b, z + c)$, for constants a, b, c.

And combinations of the above.

Line Integrals. We start with a parametrized curve $C : \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, with $a \le t \le b$. We can integrate a scalar function $f(x, y, z)$ over C or a vector field

$$\vec{F} = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$$

over C. We can calculate the line integral of a scalar function using the formula:

$$(i) \quad \int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))||\vec{r}\,'(t)||\, dt.$$

This integral is the line integral of $f(x, y, z)$ along C with respect to arclength, and a typical application might be

$$\frac{1}{\text{length}(C)} \cdot \int_C f(x, y, z)\, ds$$

is the average value of $f(x, y, z)$ along C.

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For the line integral of $\vec{F}$ along C we have

$$(ii) \int_C \vec{F} \cdot \, d\vec{r} := \int_C \vec{F} \cdot \vec{T} \, ds$$
$$= \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t) \, dt,$$

where $\vec{T}$ denotes the unit tangent vector along C.

Recall that (i) is independent of a parameterization that is 1-1 and (ii) depends upon the parameterization.

Surface integrals. We start with a parametrized surface $S : G(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, with $(u, v) \in D$ in the uv-plane.

We can integrate either a scalar function $f(x, y, z)$ or a vector field $\vec{F}(x, y, z)$ over S.
In the first case, we calculate using the formula

$$(i) \int\int_S f(x, y, z)\, dS := \int\int_D f(x(u, v), y(u, v), z(u, v))||\mathbf{T}_u \times \mathbf{T}_v||\, dA.$$

Note that the surface area of S is then

$$\int\int_S dS = \int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\, dA.$$

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For $\vec{F}$, we have

$$(ii) \int\int_S \vec{F} \cdot d\vec{S} := \int\int_S \vec{F} \cdot \vec{n}\, dS$$
$$= \int\int_D \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v \; dA,$$

where $\vec{n}$ is the unit vector field over S.

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The integral in (i) is the surface integral of $f(x, y, z)$ with respect to surface area, and a typical application might be

$$\frac{1}{\text{surface area}(C)} \cdot \int\int_S f(x, y, z)\ dS,$$

the average value of $f(x, y, z)$ over S.

If $\vec{F}$ represent a physical vector field, then the integral in (ii) yields the flux of $\vec{F}$ across S.

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Line and surface integrals can be used to define the Curl and Divergence of a vector field. You should be able to carry out these calculations for specified vector fields.

Given a vector field $\mathbf{F}(x, y, z)$ defined in a region of $\mathbb{R}^3$, a point $P \in \mathbb{R}^2$, and a unit normal vector $\mathbf{n}$, the component of the Curl of $\mathbf{F}(x, y, z)$ at P in the direction perpendicular to $\mathbf{n}$ is defined as follows:

$$\text{Curl } \mathbf{F}(P) \cdot \mathbf{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \int_C \mathbf{F} \cdot d\mathbf{r},$$

where the limit is taken over closed curves C such that $\mathbf{n}$ at P is normal to S as the areas ΔS enclosed by those curves tend to 0.

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In rectangular coordinates:

$$\textbf{Curl F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})i + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})j + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})k.$$

Given a vector field $\mathbf{F}$, the divergence of $\mathbf{F}$ at a point P measures the flux per unit volume at the point P. The divergence of $\mathbf{F}$ at P is defined by the equation:

$$\mathrm{div}\ \mathbf{F}(P) = \lim_{\mathrm{vol}(S)\to 0} \frac{1}{\mathrm{vol}(S)} \int\int_S \mathbf{F} \cdot d\mathbf{S}$$

where the limit is taken over any sequence of closed surfaces containing P whose volumes tend to 0.

How do we calculate $\mathrm{div}\ \mathbf{F}$?

Answer:The calculation depends upon the coordinate system.In rectangular coordinates,

$$\mathrm{div}\ \mathbf{F}(P) = \frac{\partial F_1}{\partial x}(P) + \frac{\partial F_2}{\partial y}(P) + \frac{\partial F_3}{\partial z}(P).$$

Important properties of line and surface integrals. Assume all of the functions below have the property that all first and second order partials are continuous.

The following are equivalent for a vector field $\vec{F}$ (a conservative vector field):

(i) $\mathrm{Curl}(\vec{F}) = 0$.

(ii) $\vec{F} = \nabla f(x, y, z)$, for some scalar function $f(x, y, z)$.

(iii) Given points P, Q in 3-space, $\int_C \vec{F} \cdot d\vec{r}$ depends only on P and Q for all curves C with initial point P and terminal point Q.

Moreover, if the conditions above hold,then $\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P)$, where the scalar function $f(x, y, z)$ satisfies $\nabla f = \vec{F}$. In particular, $\int_C \vec{F} \cdot d\vec{r} = 0$, for all closed curves C.

IMPORTANT NOTE: We did this in class for vector fields in $\mathbb{R}^2$, but this works for vector fields in $\mathbb{R}^3$ in exactly the same way.

Green's Theorem: For a closed, positively oriented curve C in the xy-plane that bounds a region R, and the vector field $\vec{F} = \langle F_1(x,y), F_2(x,y) \rangle$, we have:

$$\int_C \vec{F} \cdot \, d\vec{r} = \int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \, dA.$$

Stoke's Theorem: For a closed, positively oriented curve C in the 3-space that bounds an open surface S, and the vector field $\vec{F} = \vec{F}(x,y,z)$, we have

$$\int_C \vec{F} \cdot \, d\vec{r} = \int\int_S \mathrm{Curl}(\vec{F}) \cdot d\vec{S}.$$

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A consequence of Stoke's theorem, which can be considered a surface integral analog of the theorem characterizing conservative vector fields, is the following

Corollary. For a vector field $\mathbf{F}$ on $\mathbb{R}^3$ whose component functions have continuous first and second order partials, the following statements are equivalent.

(i) $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$.

(ii) $\int\int_S \mathbf{F} \cdot d\mathbf{S} = 0$, for all closed surfaces S.

(iii) $\int\int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_2} \mathbf{F} \cdot d\mathbf{S}$ for any two open surfaces S_1 and S_2 sharing a common boundary with the same orientation.

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Divergence Theorem: Let S be a closed, positively oriented surface that bounds the solid E in 3-space, and $\vec{F} = \vec{F}(x, y, z)$ be a vector field. Then

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int \int_E \mathrm{Div}(\vec{F})\ dV.$$

Note that if S is an open surface in 3-space bounded by the closed curve C, S is positively oriented if, in traversing C, the surface S is on the left, and the normal vectors on S point upwards in accordance with the right hand thumb rule.

A closed surface is positively oriented if its normal vectors point outward from the surface.

Note that at any point on the surface S, there are two unit normal vectors pointing in opposite directions. Any parametrization determines a system of normal vectors, and in applying any of the theorems above, one must check that the normal vectors derived from the paramterization point in the right direction.

For specific examples, you should be able to verify any of the the three theorems above. You should also be able to apply any of the theorems above to calculate one or the other of the two integrals given in each theorem.

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And finally: We have seen throughout the semester that all of our integration processes are based upon the same procedure, stated on the first day of class.

One always has a function to integrate (the *integrand*) and a domain of integration. We then described how the integration process works in all scenarios we encountered during the semester.

Namely, starting with a domain of integration and a function defined on that domain, we proceed as follows:

(i) Subdivide the domain of integration into small portions of a similar type, e.g, if the domain of integration is a solid, subdivide into smaller solids; if the domain of integration is a curve, subdivide into smaller curves.

(ii) Choose a point in each subdivision and evaluate the function at that point.

(iii) Multiply the answer in (ii) by the size of the subdivision, e.g., volume if a solid, length if a curve.

(iv) Add the quantities in (iii).

(v) Take the limit of the sums in (iv) as the size of the subdivisions tend to zero.

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The resulting numerical value depends only on the function and the underlying geometry of the domain of integration.

However, one must choose a coordinate system in which to calculate this numerical value, and employ various techniques for performing the calculation, as seen throughout the semester.